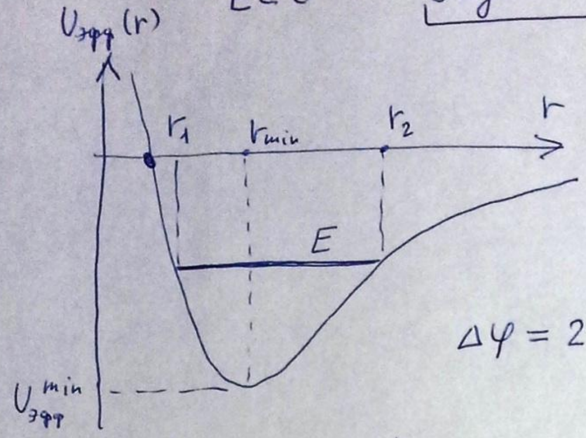
$E<0$ Задача Кеплера



$$\Delta\varphi = 2\int_{r_1}^{r_2} \frac{\frac{M}{r^2}\,dr}{\sqrt{2m(E-U)-\frac{M^2}{r^2}}} =$$

$$= 2\int_{r_1}^{r_2} \frac{\frac{M}{r^2}\,dr}{\sqrt{2m(E-U_{эфф}(r))}}$$

$$U_{эфф} = U + \frac{M^2}{2mr^2}$$

В данном случае

$$U_{эфф} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

Найдём глубину потенциальной ямы

$$U'_{эфф} = \frac{\alpha}{r^2} - \frac{M^2}{mr^3} = 0 \implies r_{min} = \frac{M^2}{\alpha m}$$

$$U_{эфф}(r_{min}) = -\frac{\alpha}{r_{min}} + \frac{M^2}{mr_{min}^3} = -\frac{\alpha^2 m}{2M^2};$$

$$\varphi = \int \frac{\frac{M}{r^2}\,dr}{\sqrt{2m(E-U(r))] - \frac{M^2}{r^2}}} + C$$

Вычислим интеграл.

$$I = \int \frac{M/r^2\,dr}{\sqrt{2mE + \frac{2\alpha m}{r} - \frac{M^2}{r^2}}} = \left| \begin{array}{l} \frac{1}{r} = \rho \\ r = \frac{1}{\rho} \\ dr = -\frac{1}{\rho^2}\,d\rho \end{array} \right| =$$

②

$$= - \int \frac{M\rho^2}{\sqrt{2mE + 2\alpha m\rho - M^2\rho^2}} \frac{d\rho}{\rho^2} =$$

$$= - \int \frac{M\,d\rho}{\sqrt{2mE + 2\alpha m\rho - M^2\rho^2}} .$$

$$2mE + 2\alpha m\rho - M^2\rho^2 = -M^2\left(\rho^2 - \frac{2\alpha m}{M^2}\rho - \frac{2mE}{M^2}\right) =$$

$$= -M^2\left(\rho^2 - 2\frac{\alpha m}{M^2}\rho + \frac{\alpha^2 m^2}{M^4} - \frac{\alpha^2 m^2}{M^4} - \frac{2mE}{M^2}\right) =$$

$$= -M^2\left[\left(\rho - \frac{\alpha m}{M^2}\right)^2 - \frac{\alpha^2 m^2}{M^4} - \frac{2mE}{M^2}\right] =$$

$$= -M^2\left(\rho - \frac{\alpha m}{M^2}\right)^2 + \frac{\alpha^2 m^2}{M^2} + 2mE$$

$$I = - \int \frac{M\,d\rho}{\sqrt{2mE + 2\alpha m\rho - M^2\rho^2}} = - \int \frac{M\,d\rho}{\sqrt{\frac{\alpha^2 m^2}{M^2} + 2mE - M^2\left(\rho - \frac{\alpha m}{M^2}\right)^2}}$$

замена $\rho - \frac{\alpha m}{M^2} = \tau$

$$I = - \int \frac{M\,d\rho}{M\sqrt{\frac{\alpha^2 m^2}{M^4} + \frac{2mE}{M^2} - \left(\rho - \frac{\alpha m}{M^2}\right)^2}} = \left\{ a^2 = \frac{\alpha^2 m^2}{M^4} + \frac{2mE}{M^2} \right.$$

$$= - \int \frac{d\tau}{\sqrt{a^2 - \tau^2}} = \arccos\frac{\tau}{a} = \arccos\left(\frac{\rho - \frac{\alpha m}{M^2}}{\sqrt{\frac{\alpha^2 m^2}{M^4} + \frac{2mE}{M^2}}}\right) + c$$

Итак

$$\varphi = \arccos\left(\frac{\frac{1}{r} - \frac{\alpha m}{M^2}}{\sqrt{\frac{\alpha^2 m^2}{M^4} + \frac{2mE}{M^2}}}\right) + C$$

или

$$\varphi = \arccos\left(\frac{\frac{M}{r} - \frac{\alpha m}{M}}{\sqrt{\frac{\alpha^2 m^2}{M^2} + 2mE}}\right) + c$$

$$\frac{M}{r} - \frac{\alpha m}{M} = \sqrt{\frac{\alpha^2 m^2}{M^2} + 2mE}\,\cos\varphi$$

$$\frac{M}{\alpha m}\cdot \Big\rangle \qquad \frac{M}{r} = \frac{\alpha m}{M} + \sqrt{\frac{\alpha^2 m^2}{M^2} + 2mE}\,\cos\varphi$$

$$\frac{M^2}{\alpha m}\,\frac{1}{r} = 1 + \frac{M}{\alpha m}\sqrt{\frac{\alpha^2 m^2}{M^2} + 2mE}\,\cos\varphi$$

$$p = \frac{M^2}{\alpha m}\ ; \qquad e = \sqrt{1 + \frac{2mEM^2}{\alpha^2 m^2}} = \sqrt{1 + \frac{2EM^2}{m\alpha^2}} < 1$$

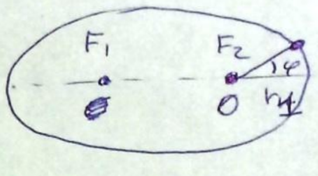
при $E<0$

e – эксцентриситет

p – параметр

$$\boxed{\frac{p}{r} = 1 + e\cos\varphi}$$

или $r(\varphi) = \dfrac{p}{1+e\cos\varphi}$



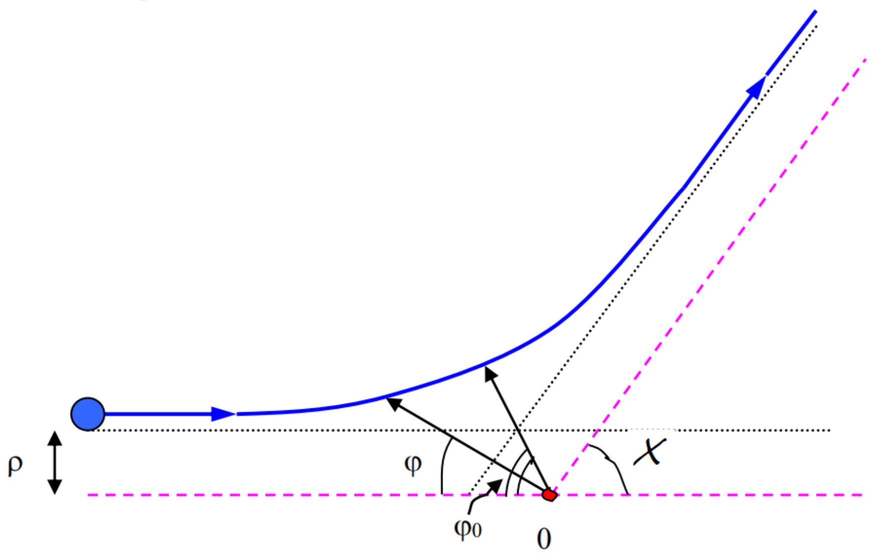
Задача рассеяния

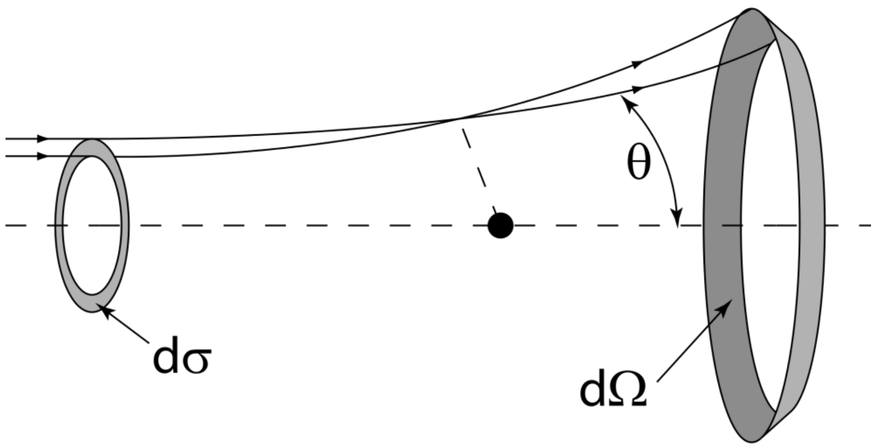
$$\varphi = \int_{r_1}^{\infty} \frac{M/r^2}{\sqrt{2m\,(E-U) - \frac{M^2}{r^2}}}$$

$$E = \frac{m v_\infty^2}{2} \qquad M = m\rho v_\infty$$

$$\varphi = \int_{r_1}^{\infty} \frac{\frac{\rho\, dr}{r^2}}{\sqrt{1 - \frac{2}{m v_\infty^2}U - \frac{\rho^2}{r^2}}}$$

$$d\sigma = \frac{dN}{n}$$


ρ
φ
φ0
0
χ

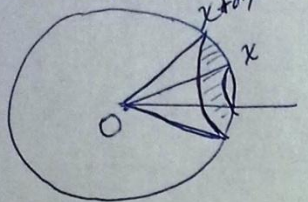

θ
dσ
dΩ

$dN = 2\pi\rho\, d\rho \cdot n$; $d\sigma = 2\pi\rho\, d\rho$

Пусть χ — монотонно зависит от ρ

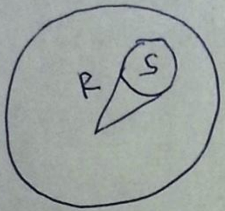
$$d\sigma = 2\pi\rho(\chi)\left|\frac{d\rho(\chi)}{d\chi}\right| d\chi$$

Телесный угол



$$d\Omega = 2\pi \sin\chi\, d\chi$$

$$d\sigma = \frac{\rho(\chi)}{\sin\chi}\left|\frac{d\rho}{d\chi}\right| d\Omega$$



$$\Omega = \frac{S}{R^2}$$

Формула Резерфорда

Рассмотрим рассеяние в поле $U = \frac{\alpha}{r}$

Вычислим интеграл $$I = \int \frac{\frac{\rho}{r^2}\, dr}{\sqrt{1 - \frac{2}{mv^2}\frac{\alpha}{r} - \frac{\rho^2}{r^2}}} = \left| \begin{array}{l} \frac{1}{r} = \tau \\ r = \frac{1}{\tau} \\ dr = -\frac{d\tau}{\tau^2} \end{array} \right| =$$

$$= -\int \frac{d\tau}{\sqrt{\frac{1}{\rho^2} - \frac{2\alpha}{mv^2\rho^2}\tau - \tau^2}} ;$$

$$\frac{1}{\rho^2} - \frac{2\alpha}{mv^2\rho^2}\tau - \tau^2 = -\left(\tau^2 + \frac{2\alpha}{mv^2\rho^2}\tau + \frac{\alpha^2}{m^2v^4\rho^4} - \frac{\alpha^2}{m^2v^4\rho^4} - \frac{1}{\rho^2}\right) =$$

$$= -\left(\tau + \frac{\alpha}{mv^2\rho^2}\right)^2 + \frac{\alpha^2}{m^2v^4\rho^4} + \frac{1}{\rho^2}$$

$$I = -\int \frac{d\tau}{\sqrt{\left(\frac{1}{\rho^2} + \frac{\alpha^2}{m^2 v^4 \rho^4}\right) - \left(\tau + \frac{\alpha}{m v^2 \rho^2}\right)^2}} =$$

$$= \arccos\left(\frac{\tau + \frac{\alpha}{m v^2 \rho^2}}{\sqrt{\frac{1}{\rho^2} + \frac{\alpha^2}{m^2 v^4 \rho^4}}}\right) = \arccos\left(\frac{\frac{1}{r} + \frac{\alpha}{m v^2 \rho^2}}{\sqrt{\frac{1}{\rho^2} + \frac{\alpha^2}{m^2 v^4 \rho^4}}}\right)$$

$$\varphi_0 = \arccos\left(\frac{\frac{1}{r} + \frac{\alpha}{m v^2 \rho^2}}{\sqrt{\frac{1}{\rho^2} + \frac{\alpha^2}{m^2 v^4 \rho^4}}}\right)\Bigg|_{r_{min}}^{\infty} = \arccos\left(\frac{\frac{\alpha}{m v^2 \rho}}{\sqrt{1 + \frac{\alpha^2}{m^2 \rho^2 v^4}}}\right),$$

т.к. $$\frac{1}{r_{min}} + \frac{\alpha}{m v^2 \rho^2} = \sqrt{\frac{1}{\rho^2} + \frac{\alpha^2}{m^2 v^4 \rho^4}}$$

$$\cos^2\varphi_0 = \frac{\frac{\alpha^2}{m^2 \rho^2 v^4}}{1 + \frac{\alpha^2}{m^2 \rho^2 v^4}} \qquad \Bigg\} \quad 1 + tg^2\varphi_0 = \frac{1}{\cos^2\varphi_0}$$

$$\frac{1}{1 + tg^2\varphi_0} = \frac{\frac{\alpha^2}{m^2 \rho^2 v^4}}{1 + \frac{\alpha^2}{m^2 \rho^2 v^4}}; \quad 1 + tg^2\varphi_0 = \frac{1 + \frac{\alpha^2}{m^2 \rho^2 v^4}}{\frac{\alpha^2}{m^2 \rho^2 v^4}}$$

$$tg^2\varphi_0 = \frac{m^2 \rho^2 v^4}{\alpha^2} \implies \rho^2 = \frac{\alpha^2}{m^2 \cancel{\rho} v^4}\, tg^2\varphi_0$$

$$\varphi_0 = \frac{\pi}{2} - \frac{\chi}{2}$$

$$\rho^2 = \frac{\alpha^2}{m^2 v^4}\, tg^2\left(\frac{\pi}{2} - \frac{\chi}{2}\right); \quad \rho^2 = \frac{\alpha^2}{m^2 v^4}\, ctg^2\frac{\chi}{2}$$

$$2\rho\, d\rho = \frac{\alpha^2}{m^2 v^4}\, 2\, ctg\frac{\chi}{2} \cdot \left|\frac{-1}{\sin^2\frac{\chi}{2}}\right| \cdot \frac{1}{2}\, d\chi$$

$$d\sigma = 2\pi\rho\, d\rho = \pi \frac{\alpha^2}{m^2 v^4} \frac{\cos\chi/2}{\sin\chi/2} \cdot \frac{d\chi}{\sin^2\chi/2} =$$

$$= \pi\left(\frac{\alpha}{m v^2}\right)^2 \frac{\cos\frac{\chi}{2}\, d\chi}{\sin^3\chi/2}$$

учитывая, что

$$d\Omega = 2\pi \sin\chi \, d\chi$$

$$\Rightarrow \quad d\chi = \frac{d\Omega}{2\pi \sin\chi}$$

$$d\sigma = \pi \left(\frac{\alpha}{mv^2}\right)^2 \frac{\cos\frac{\chi}{2}}{\sin^3 \chi/2} \frac{d\Omega}{2 \cdot 2\pi \sin\chi/2 \cos\chi/2} =$$

$$= \boxed{d\sigma = \left(\frac{\alpha}{2mv^2}\right)^2 \frac{d\Omega}{\sin^4 \chi/2}}$$